Vieta's Formulas

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November 27, 2017

Introduction

Vieta's formulas are several formulas that relate the coefficients of a polynomial to its roots. For a quadratic $ax^2 + bx + c$ with roots r_1 and r_2, Vieta's formulas state that

$$r_1 + r_2 = -\frac{b}{a}, \quad r_1 r_2 = \frac{c}{a}.$$

This can be shown by noting that $ax^2 + bx + c = a(x - r_1)(x - r_2)$, expanding the right hand side, then comparing coefficients. For a cubic polynomial $ax^3 + bx^2 + cx + d$ with roots r_1, r_2, and r_3, we have

$$r_1 + r_2 + r_3 = -\frac{b}{a}, \quad r_1r_2 + r_2r_3 + r_3r_1 = \frac{c}{a}, \quad r_1r_2r_3 = -\frac{d}{a}.$$

Finally, Vieta's formulas can be generalized to any polynomial. Given an nth degree polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with roots $r_1, r_2, \ldots, r_n$, Vieta's formulas state that

$$\begin{cases} r_1 + r_2 + r_3 + \cdots + r_n = -\frac{a_{n-1}}{a_n} \\ (r_1r_2 + r_1r_3 + \cdots + r_1r_n) + (r_2r_3 + r_2r_4 + \cdots + r_2r_n) + \cdots + r_{n-1}r_n = \frac{a_{n-2}}{a_n} \\ \vdots \\ r_1r_2r_3 \cdots r_n = (-1)^n \frac{a_0}{a_n} \end{cases}$$

Note that the sign alternates between positive and negative. Also, the n roots don't have to be real - the formulas hold for complex roots too.

Examples

1. Integers x and y satisfy $xy + x + y = 71$ and $x^2y + xy^2 = 880$. Find x and y.
2. Let n be a positive integer, and for $1 \leq k \leq n$, let s_k be the sum of the $\binom{n}{k}$ products of the numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$ taken k at a time. For example,

$$s_1 = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \text{and} \quad s_2 = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} + \cdots \frac{1}{n-1} \cdot \frac{1}{n}$$

   Find $s_1 + s_2 + s_3 + \cdots + s_n$.

Problems

1. Find all triples of complex numbers satisfying

$$\begin{cases} a+b+c=0 \\ ab+bc+ca=0 \\ abc=0 \end{cases}$$

2. (AIME I 2005) The equation $2^{333x-2}+2^{111x+2}=2^{222x+1}+1$ has three real roots. Find their sum.

3. (AIME I 2001) Find the sum of all the roots, real and non-real, of the equation $x^{2001}+\left(\frac{1}{2}-x\right)^{2001}=0$, given that there are no multiple roots.

4. Let a, b, and c be real numbers such that $a+b+c>0$, $ab+bc+ca>0$, and $abc>0$. Prove that a, b, and c are all positive.

5. (CMO 1996) If α, β, γ are the roots of $x^3-x-1=0$, compute $\frac{1+\alpha}{1-\alpha}+\frac{1+\beta}{1-\beta}+\frac{1+\gamma}{1-\gamma}$.

6. (HMMT November 2016 Guts Round) Let r_1, r_2, r_3, r_4 be the four roots of polynomial $x^4-4x^3+8x^2-7x+3$. Find the value of

$$\frac{r_1^2}{r_2^2+r_3^2+r_4^2}+\frac{r_2^2}{r_1^2+r_3^2+r_4^2}+\frac{r_3^2}{r_1^2+r_2^2+r_4^2}+\frac{r_4^2}{r_1^2+r_2^2+r_3^2}$$

7. Let r, s, and t be the three roots of the equation

$$8x^3+1001x+2008=0.$$

Find $(r+s)^3+(s+t)^3+(t+r)^3$.

8. Let z_1, z_2, and z_3 be three complex numbers such that $z_1+z_2+z_3=\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}$ and $z_1z_2z_3=1$. Show that at least one of the z_i's must be 1.