Tangent Circles

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Useful Techniques

Homothety

- If two circles are tangent at T, then there is a homothety at T which sends one circle to the other.
- This is a negative homothety when the circles are externally tangent, and a positive homothety when they are internally tangent.
- If you have points A, B, C on the first circle, you can use the homothety to get new points A', B', C' on the second circle.
- Conversely, if you have know that triangles ABC and $A'B'C'$ are homothetic, with the center of homothety on one of their circumcircles, then their circumcircles are tangent to each other at T.

Inversion

- If two circles are tangent at T, an inversion centered at T will turn the circles into parallel lines.
- An inversion centered on one of the circles will turn the circle-circle tangency into a circle-line tangency.

Collinear Centers

- Let O_1 and O_2 be the centers of two circles which are tangent at T. Then O_1, O_2, T are collinear.
- O_1T and O_2T are the radii of the circles. The length of O_1O_2 is either the sum or difference of the radii, depending on whether the circles are externally or internally tangent.
- Conversely, if you know the radii and distance between the centers, you can prove that they are tangent. (might require some sort of length bash)

Common Tangent

- Suppose T is a point on circles ω_1 and ω_2. Then the circles are tangent iff the tangent to ω_1 at T is the same line as the tangent to ω_2 at T.
- The common tangent is also the radical axis of ω_1 and ω_2, so try using Power of a Point centered at some point on the tangent line.

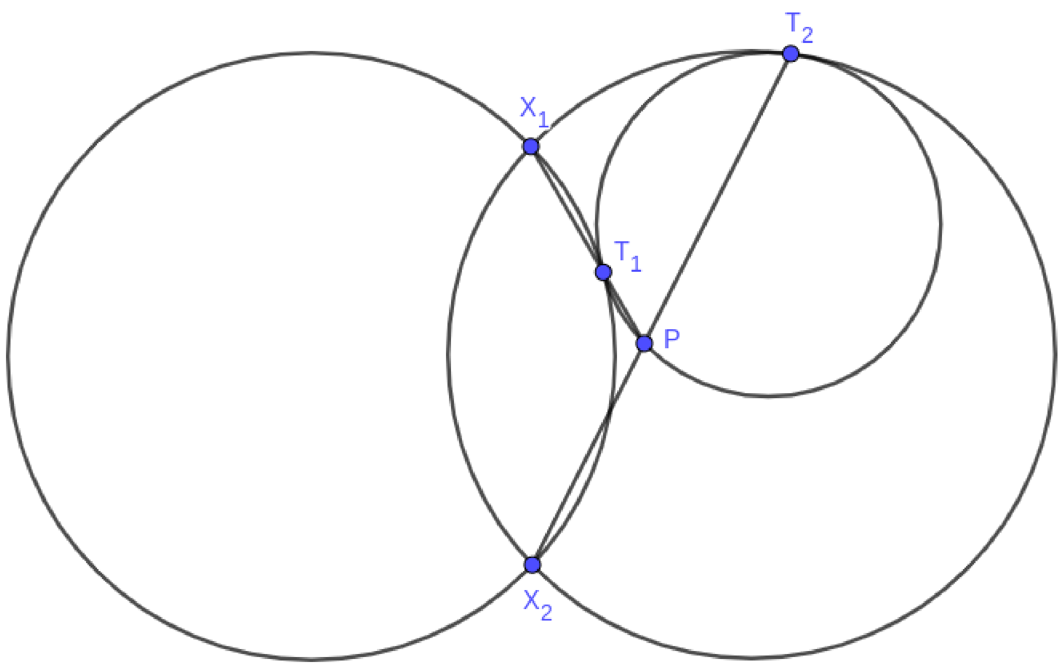
Mixtilinear Incircle

- The A-mixtilinear incircle of triangle ABC is the unique circle tangent to AB, AC, and internally tangent to the circumcircle of ABC.
- Evan Chen's handout: https://web.evanchen.cc/handouts/Mixt-GeoGuessr/Mixt-GeoGuessr.pdf
- Section 2 of Yufei Zhao's handout: https://yufeizhao.com/olympiad/imo2008/zhao-circles.pdf

Example Problems

Example 1 (EGMO 2016 P4)

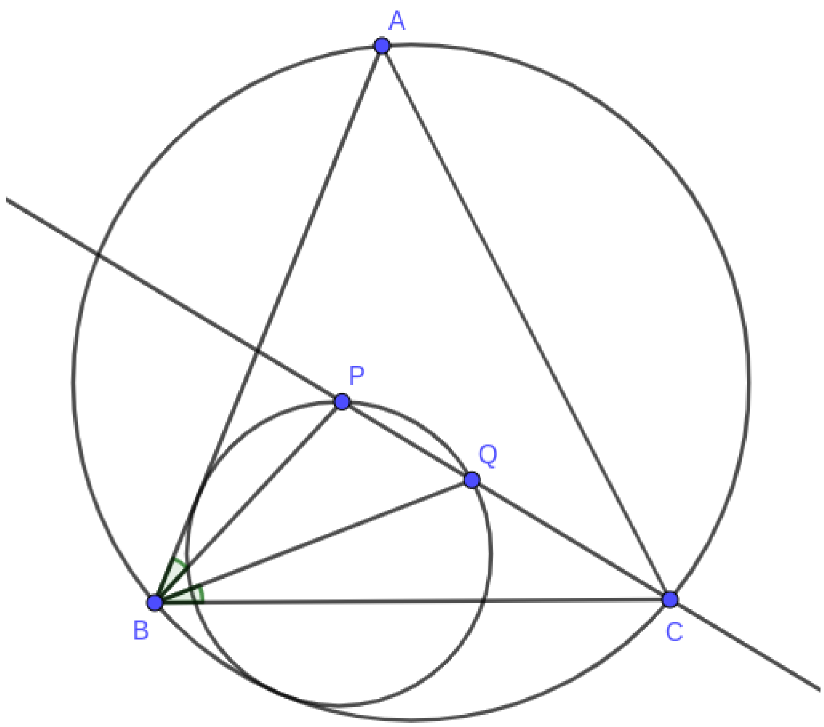
Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.



Example 2 (EGMO 2018 P5)

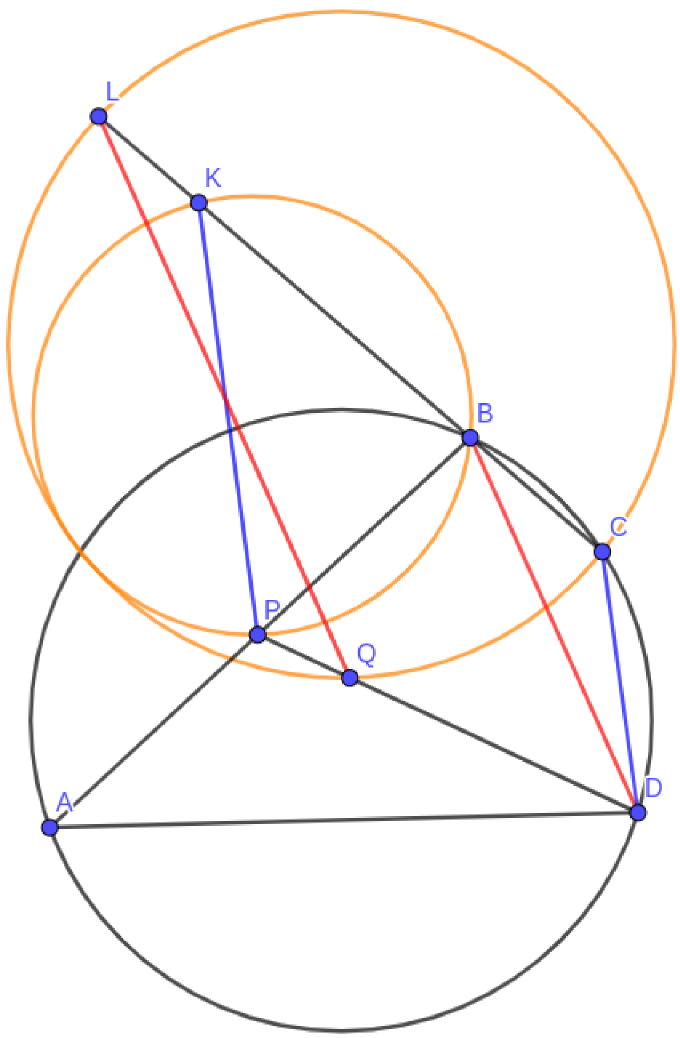
Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q.

Prove that $\angle ABP = \angle QBC$.



Example 3 (RMM 2018 P1)

Let $ABCD$ be a cyclic quadrilateral and let P be a point on the side AB. The diagonal AC meets the segment DP at Q. The line through P parallel to CD meets the extension of the side CB beyond B at K. The line through Q parallel to BD meets the extension of the side CB beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent.



Practice Problems

1. **(ELMO 2012 P1)** In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

2. **(EGMO 2013 P5)** Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q.

   Prove that $\angle ACP = \angle QCB$.

3. **(ISL 2020 G3)** Let $ABCD$ be a convex quadrilateral with $\angle ABC > 90$, $\angle CDA > 90$ and $\angle DAB = \angle BCD$. Denote by E and F the reflections of A in lines BC and CD, respectively. Suppose that the segments AE and AF meet the line BD at K and L, respectively. Prove that the circumcircles of triangles BEK and DFL are tangent to each other.

4. **(ISL 2011 G4)** Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

5. Non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3. For each i, M_i is the midpoint of a_i, T_i is where a_i meets the incircle of $A_1A_2A_3$, and S_i is the reflection of T_i over the angle bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent.

6. **(ISL 2017 G4)** In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

7. **(ELMO 2016 P6)** Elmo is now learning olympiad geometry. In triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90°$. Finally, let γ be the circumcircle of $\triangle AST$.

   (a) Help Elmo show that γ is tangent to the circumcircle of $\triangle ABC$.

   (b) Help Elmo show that γ is tangent to the incircle of $\triangle ABC$.

8. **(ISL 2020 G6)** Let ABC be a triangle with $AB < AC$, incenter I, and A excenter I_A. The incircle meets BC at D. Define $E = AD \cap BI_A$, $F = AD \cap CI_A$. Show that the circumcircle of $\triangle AID$ and $\triangle I_AEF$ are tangent to each other.

9. **(RMM 2018 P6)** Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.