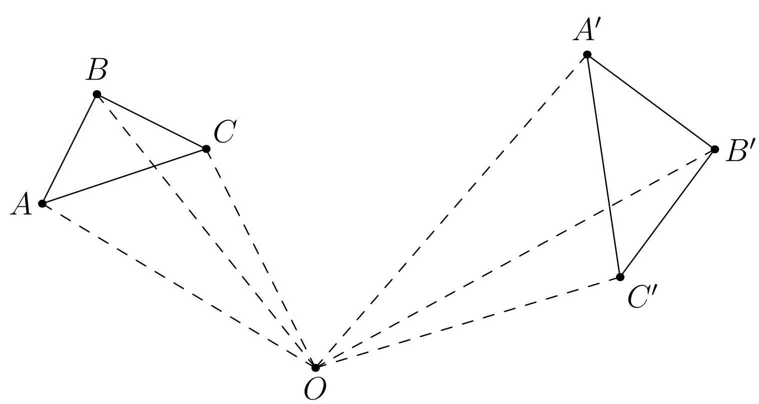
Spiral Similarity

Howard Halim

January 10, 2020

Definitions and Basics

A *spiral similarity* is a transformation centered at a point O. It consists of a rotation around O, followed by a homothety at O.



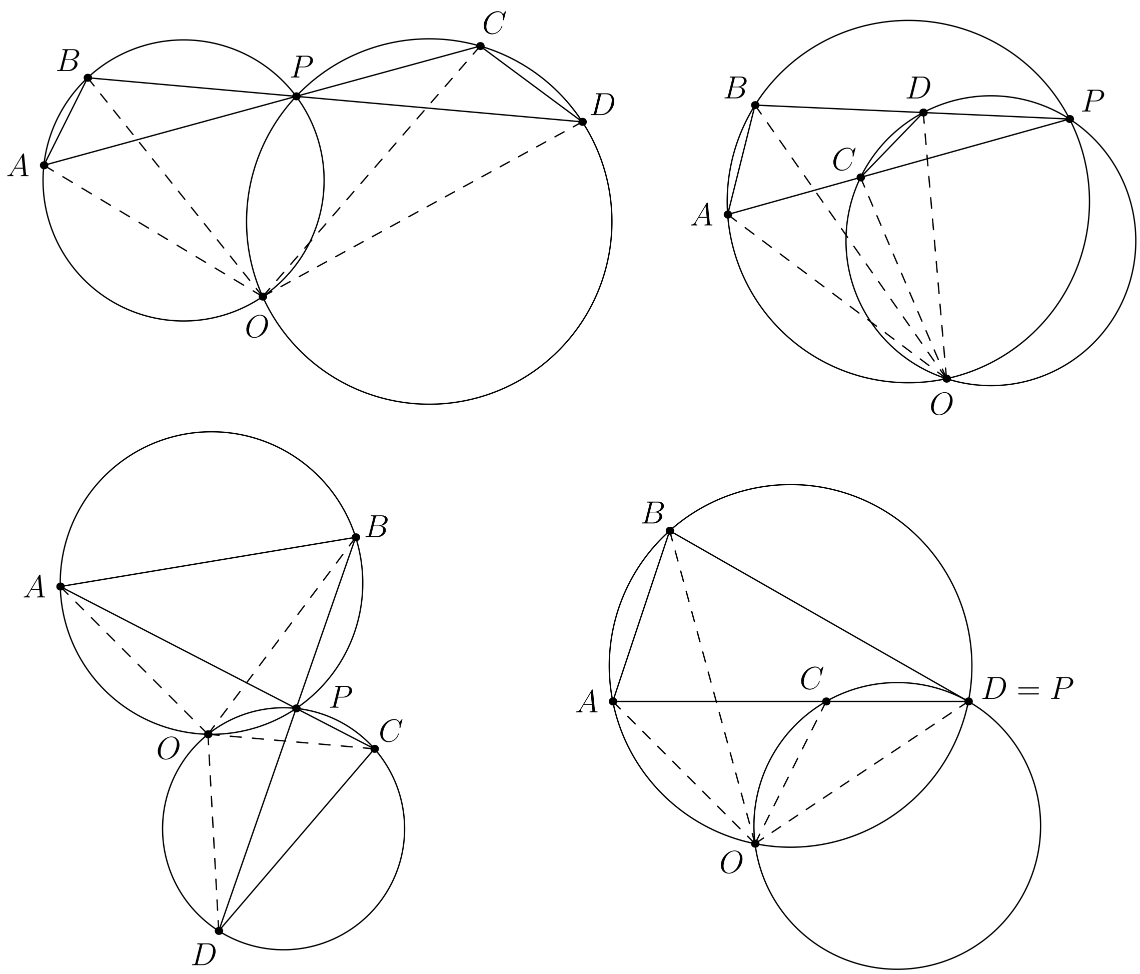
Spiral similarities preserve similarities (hence the name). ABC is similar to $A'B'C'$ in the diagram above. In fact, $ABCO$ is similar to $A'B'C'O$. Any point relative to A, B, and C will be sent to the corresponding point relative to A', B', and C'. For example, the midpoint of AB goes to midpoint of $A'B'$, the orthocenter of ABC goes to orthocenter of $A'B'C'$, etc.

It's clear that given any 3 points A, B, and O, there's a unique spiral similarity at O which sends A to B. Just rotate by angle $\angle AOB$, then scale by $\dfrac{BO}{AO}$.

A less obvious fact: Given any 4 points A, B, C, D, there is a unique spiral similarity sending AB to CD.

Important Lemmas

Lemma 1: Let P be the intersection of lines AC and BD. Let O be the second intersection of circles (PAB) and (PCD). Then O is the center of the spiral similarity which sends AB to CD.



Proof: Directed angle chase:

$$\measuredangle ABO = \measuredangle APO = \measuredangle CPO = \measuredangle CDO.$$

Similarly, $\measuredangle BAO = \measuredangle DCO$, so $\triangle ABO \sim \triangle CDO$ by AA similarity. Therefore, O is the center of the spiral similarity.

Lemma 2: If there is a spiral similarity at O which sends AB to CD, then there is also a spiral similarity at O which sends AC to BD.

Proof: Since $\triangle ABO \sim \triangle CDO$, we have

$$\angle AOB = \angle COD \qquad \text{and} \qquad \frac{AO}{BO} = \frac{CO}{DO}$$

If we rotate by $\angle AOB$ and scale by $\frac{BO}{AO}$, this spiral similarity will send A to B. But this is the same angle and scale factor needed to send C to D. Therefore, this second spiral similarity sends AC to BD.

Examples

Example 1: In triangle ABC, let P be the midpoint of arc BAC on the circumcircle. Let X and Y be points on AB and AC respectively, such that $APXY$ is cyclic. Prove that $BX = CY$.

Proof: By Lemma 1, P is the center of the spiral similarity sending BC to XY. By Lemma 2, P is also the center of the spiral similarity sending BX to CY. That means triangles PBX and PCY are similar. But since P is on the perpendicular bisector of BC, $PB = PC$, so triangles PBX and PCY are actually congruent! In other words, the spiral similarity $BX \to CY$ is just a rotation (no scaling). Therefore, $BX = CY$ because the line segments are rotations of one another.

Example 2: Let $A_1A_2A_3$ and $B_1B_2B_3$ be similar triangles. Let G_1, G_2, G_3 be the centroids of triangles $A_1B_2B_3$, $A_2B_3B_1$, and $A_3B_1B_2$, respectively. Prove that $G_1G_2G_3$ is similar to $A_1A_2A_3$ and $B_1B_2B_3$.

Proof: Let C_1, C_2, C_3 be the midpoints of B_2B_3, B_3B_1, B_1B_2. Since $\triangle A_1A_2A_3 \sim \triangle B_1B_2B_3 \sim \triangle C_1C_2C_3$, triangles $A_1A_2A_3$ and $C_1C_2C_3$ have a spiral center O. But it is a well-known fact about the centroid that $A_iG_i = 2G_iC_i$ for all i. Therefore, the spiral similarity sending A_1C_1 to A_2C_2 also sends G_1 to G_2 (the same argument works for $A_1G_1C_1 \to A_3G_3C_3$). This proves that

$$A_1G_1C_1O \sim A_2G_2C_2O \sim A_3G_3C_3O.$$

By Lemma 2, there is a spiral similarity from O which takes $A_1A_2A_3$ to $G_1G_2G_3$, so the triangles are similar, as desired.

Example 3: Let $\triangle ABC$ be an acute scalene triangle, O be its circumcenter and D the midpoint of side BC. The circle with diameter AD meets sides AB and AC again at points E and F, respectively. The line through D parallel to AO meets EF at M. Show that $EM = MF$.

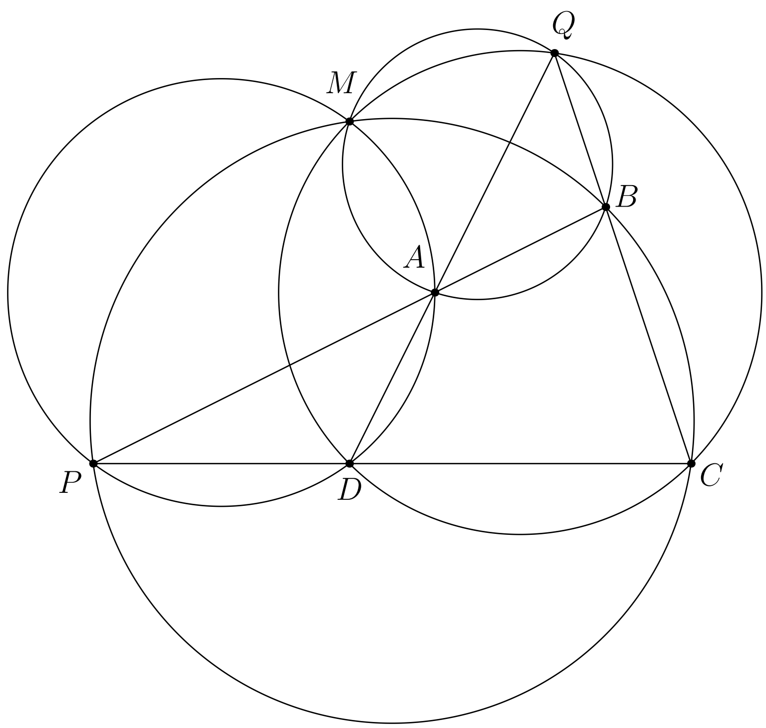
Proof: First, redefine M to be the midpoint of EF, and we'll prove that $DM \| AO$. Let BB' and CC' be the altitudes of ABC, and let D' be the midpoint of $B'C'$. Since $BCB'C'$ is cyclic with D as its circumcenter, $DB = DC = DB' = DC'$.

Therefore, triangles DBC', DCB', and $DB'C'$ are all isosceles. That means the projections of D onto the bases of the respective isosceles triangles are midpoints (E is the midpoint of BC', F is the midpoint of CB', and $DD' \perp B'C'$ since D' is the midpoint of $B'C'$).

Now, we prove that D, M, and D' are collinear. Let the circle with diameter AD intersect the circumcircle at P. By Lemma 1, P is the center of the spiral similarity which sends BC to EF. It is also the center of the spiral similarity $BE \to CF$. But $BEC' \sim CFB'$, so it must also send C' to B'. Apply Lemma 2 to get spiral similarities at P which send BC to EF and $C'B'$. These must send D to the respective midpoints M and D'. Applying Lemma 2 again tells us that there's a spiral similarity $BEC' \to DMD'$, so M is on DD' (in fact, this proves M is the midpoint of DD').

Since $DM = DD' \perp B'C'$ and $AO \perp B'C'$, $DM \| AO$.

Miquel Point



Fact 1. (Existence of the Miquel Point) Let $ABCD$ be a quadrilateral, $P = AB \cap CD$, $Q = AD \cap BC$. Then the circumcircles of ADP, BCP, ABQ, and CDQ concur at a point M. This point is called the *Miquel Point* of quadrilateral $ABCD$.

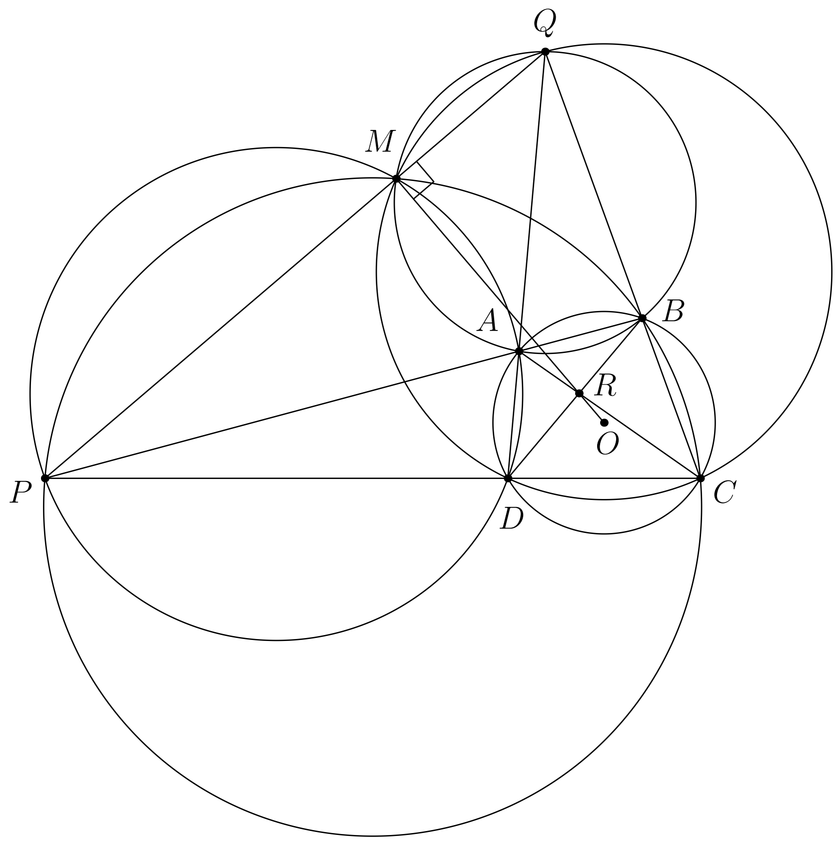
Fact 2. M is the center of 6 spiral similarities: $AB \to DC$, $AD \to BC$, $DP \to QB$, $BP \to QD$, $AP \to QC$, and $CP \to QA$.

Fact 3. The circumcenters of ADP, BCP, ABQ, and CDQ are concyclic with M.

Fact 4. The projections from M onto AB, BC, CD, DA are collinear.

Miquel Points of Cyclic Quadrilaterals

When we restrict the configuration to Miquel points of cyclic quadrilaterals, we get many more properties!



Let $ABCD$ be a cyclic quadrilateral with circumcenter O and Miquel point M. Let $P = AB \cap CD$, $Q = AD \cap BC$, $R = AC \cap BD$. Then the following are true:

Fact 5. M lies on line PQ.

Fact 6. The circumcircles of AOC and BOD pass through M.

Fact 7. M, O, R are collinear.

Fact 8. Lines MA and MC are symmetric wrt line MO. Similarly, lines MB and MD are also symmetric.

Fact 9. The circles with diameters PO and QO pass through M.

Fact 10. MO is perpendicular to PQ.

Fact 11. M and R are inverses wrt the circumcircle of $ABCD$.

Fact 12. PQR is self-polar wrt the circumcircle of $ABCD$.

Fact 13. O is the orthocenter of PQR.

Problems

1. (Euclid 2014) ABC and CDE are equilateral triangles, with B, C, D collinear, and A and E on the same side of line BC. Let M be the midpoint of BE, and N be the midpoint of AD. Prove that $\triangle MNC$ is equilateral.

2. (IMO 1985) A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.

3. (Russia 1995) Quadrilateral $ACDB$ is inscribed in a semicircle with diameter AB and point O is the midpoint of AB. Let K be the intersection of the circumcircles of AOC and BOD. Lines AB and CD intersect at M. Prove that $\angle OKM = 90°$.

4. (China 1992) Convex quadrilateral $ABCD$ is inscribed in circle ω with center O. Diagonals AC and BD meet at P. The circumcircles of triangles ABP and CDP meet at P and Q. Assume that points O, P, and Q are distinct. Prove that $\angle OQP = 90°$.

5. (Russia 1999) A circle through vertices A and B of a triangle ABC meets side BC again at D. A circle through B and C meets side AB at E and the first circle again at F. Prove that if points A, E, D, C lie on a circle with center O, then $\angle BFO = 90°$.

6. (Fermat Point) Let ABC be a triangle, and construct equilateral triangles BCD, CAE, ABF outside of ABC. Prove that AD, BE, CF are concurrent.

7. (USAMO 2013) In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.

8. (USAMO 2006) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

9. In triangle ABC, points E and F lie on AB and AC respectively, such that $BE = CF$. Prove that the radical axis of the circumcircles of ABF snd ACE bisects $\angle BAC$.

10. (APMO 2017) Let ABC be a triangle with $AB < AC$. Let D be the intersection point of the internal bisector of angle BAC and the circumcircle of ABC. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of triangle ADZ.

11. (Brazil 2011) Let ABC be an acute triangle and H is orthocenter. Let D be the intersection of BH and AC and E be the intersection of CH and AB. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on BC.

12. (USA TSTST 2012) Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

13. (USA TST 2007) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lie on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

14. (ISL 2016) Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.

15. (USAMO 2018) In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90°$.

16. (ISL 2015) Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

17. (CMO 2007) Let the incircle of triangle ABC touch sides BC, CA, and AB at D, E, and F, respectively. Let $\omega, \omega_1, \omega_2$, and ω_3 denote the circumcircles of triangle ABC, AEF, BDF, and CDE respectively. Let ω and ω_1 intersect at A and P, ω and ω_2 intersect at B and Q, ω and ω_3 intersect at C and R. Show that PD, QE and RF are concurrent.

18. (ISL 2005) Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

19. (IMO 2013) Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

20. (ISL 2006) Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.