Power of a Point

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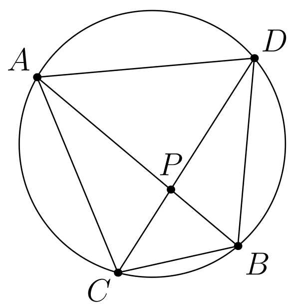
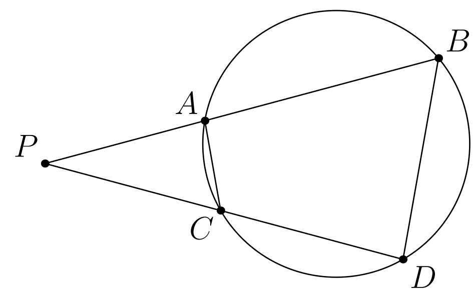
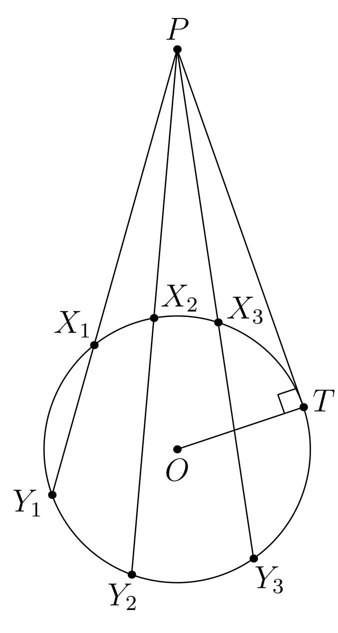
Power of a Point: Let A, B, C, and D be four points on a circle. Let P be the intersection of lines AB and CD. Then,

$$PA \cdot PB = PC \cdot PD.$$





Another way of stating this theorem is: Given a point P and a circle ω, let a line passing through P intersect ω at X and Y. Then the product $PX \cdot PY$ has constant value, no matter which line we choose. This constant value is called the power of P with respect to ω, and is written as $\text{Pow}(P, \omega)$.



A special case is when the line passing through P is tangent to the circle at a point T. Both X and Y are the point of tangency ($X = Y = T$), so $\text{Pow}(P, \omega) = PX \cdot PY = PT^2$. From this, we see that $\text{Pow}(P, \omega) = PO^2 - r^2$, where O is the center of the circle, and r is the radius. (Note: PX and PY are directed lengths.) $\text{Pow}(P, \omega)$ is positive when P is outside of the circle, negative when P is inside the circle, and 0 when P is on the circle.

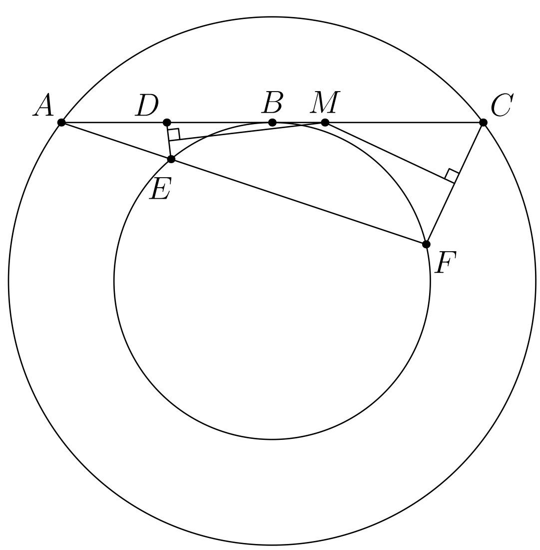
Converse of Power of a Point: Let A, B, C, D be four distinct points, and P be the intersection of lines AB and CD. If $PA \cdot PB = PC \cdot PD$, then $ABCD$ is cyclic (lengths are directed).

Power of a Point Problems

1. (HMMT 2007) A, B, C, and D are points on a circle, and segments $\overline{AC}$ and $\overline{BD}$ intersect at P, such that $AP = 8$, $PC = 1$, and $BD = 6$. Find BP, given that $BP < DP$.

2. Let ABC be a triangle with D on AB and E on AC such that $DE \perp AB$. If $BE = 4$ and $AD = BD = CD = 3$, find the length of CE.

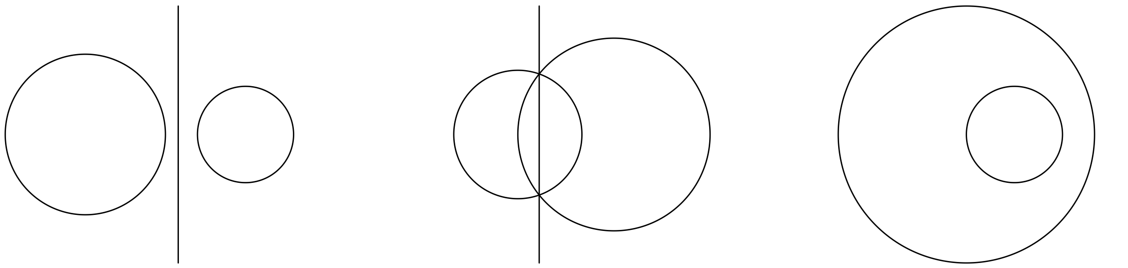
3. (USAMO 1998) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.



Radical Axis

Let ω_1 and ω_2 be two circles with centers O_1 and O_2, and radii r_1 and r_2. The radical axis of these two circles is the set of points with equal power with respect to both circles - i.e. the locus of points P such that

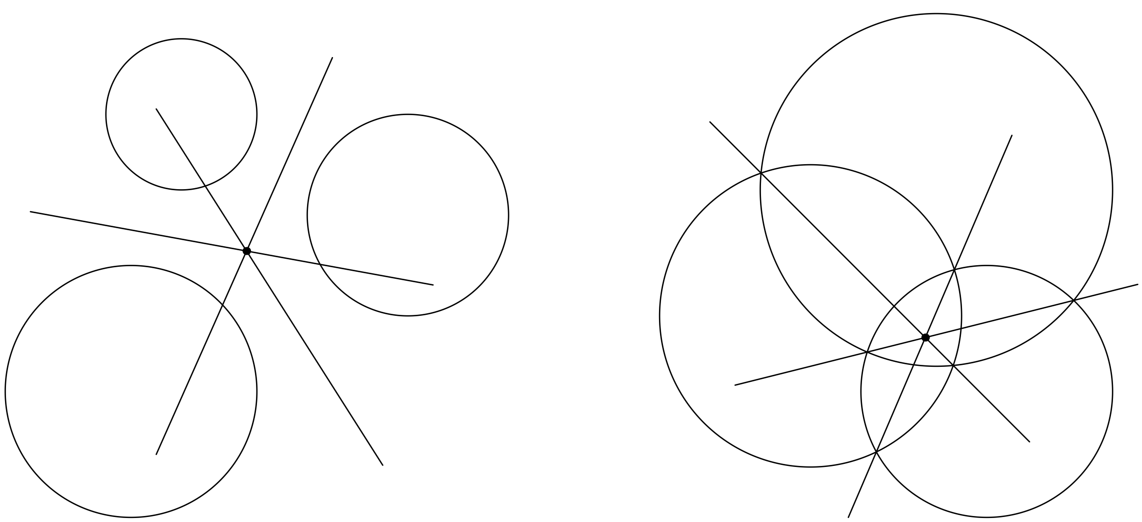
$$\mathrm{Pow}(P, \omega_1) = \mathrm{Pow}(P, \omega_2).$$



The radical axis is a line perpendicular to the line connecting the centers of the circles. If the circles intersect, their radical axis is the line passing through the intersection points (see the middle diagram).

Radical Center

Radical Axis Theorem: Let ω_1, ω_2 and ω_3 be three circles. Let $l_1, 2$ be the radical axis of ω_1 and ω_2, $l_2, 3$ be the radical axis of ω_2 and ω_3, and $l_3, 1$ be the radical axis of ω_3 and ω_1. Then $l_1, 2$, $l_2, 3$, and $l_3, 1$ are either concurrent or all parallel. If they are concurrent (which happens most of the time), the point of concurrency is called the radical center of the three circles.



Radical Axis Problems

1. Two circles intersect at A and B, and a line is tangent to the circles at C and D. Prove that AB bisects CD.

2. Let A, B, C be three points on a circle Λ with $AB = BC$. Let the tangents at A and B meet at D. Let DC meet Λ again at E. Prove that the line AE bisects segment BD.

3. (USAMO 1990) An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

4. Let ABC be an acute triangle. The points M and N are taken on the sides AB and AC, respectively. The circles with diameters BN and CM intersect at points P and Q. Prove that P, Q, and the orthocenter H are collinear.

5. (USAMO 1992) Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

Challenge Problems

1. Let ABC be a triangle and let D and E be points on the sides AB and AC, respectively, such that DE is parallel to BC. Let P be any point interior to triangle ADE, and let F and G be the intersections of DE with the lines BP and CP, respectively. Let Q be the second intersection point of the circumcircles of triangles PDG and PFE. Prove that the points A, P, and Q are collinear.

2. (IMO 1995) Let A, B, C, and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, and XY are concurrent.

3. (IMO 2008) Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects line BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

   Prove that six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.