Inversion Problems

1. (USAMO 1993 P2) Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

2. Let $\omega_1, \omega_2, \omega_3, \omega_4$ be circles such that ω_i and ω_{i+1} are externally tangent at a point A_i for all $i = \{1,2,3,4\}$ (indices taken mod 4). Prove that A_1, A_2, A_3, A_4 are concyclic.

3. (Brazil 2009 P5) Let ABC be a triangle and O its circumcenter. Lines AB and AC meet the circumcircle of OBC again in $B_1 \neq B$ and $C_1 \neq C$, respectively, lines BA and BC meet the circumcircle of OAC again in $A_2 \neq A$ and $C_2 \neq C$, respectively, and lines CA and CB meet the circumcircle of OAB in $A_3 \neq A$ and $B_3 \neq B$, respectively. Prove that lines A_2A_3, B_1B_3 and C_1C_2 have a common point.

4. (BAMO 2008 P4) A point D lies inside triangle ABC. Let A_1, B_1, C_1 be the second intersection points of the lines AD, BD, and CD with the circumcircles of BDC, CDA, and ADB, respectively. Prove that
$$\frac{AD}{AA_1} + \frac{BD}{BA_1} + \frac{CD}{CC_1} = 1.$$

5. (NIMO Winter 2014 P7) Let ABC be a triangle and let Q be a point such that $\overline{AB} \perp \overline{QB}$ and $\overline{AC} \perp \overline{QC}$. A circle with center I is inscribed in $\triangle ABC$, and is tangent to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ at points D, E, and F, respectively. If ray QI intersects $\overline{EF}$ at P, prove that $\overline{DP} \perp \overline{EF}$.

6. (Inversion Distance Formula) Let A, B, and O be 3 distinct points. An inversion centered at O with radius r sends A to A' and B to B'. Prove that
$$A'B' = \frac{r^2}{AO \cdot BO} \cdot AB.$$

7. (Ptolemy's Inequality) Let A, B, C, D be point on the plane, no 3 collinear. Prove that
$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD$$
with equality iff A, B, C, D are concyclic.

8. (USAMO 2009 P5) Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to $\overline{AB}$ intersects $\overline{BD}$ and $\overline{BC}$ at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if $\overline{BG}$ bisects $\angle CBD$.

9. (ELMO 2018 P3) Let A be a point in the plane, and ℓ a line not passing through A. Evan does not have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is [i]not[/i] marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

   (i) Can Evan construct the reflection of A over ℓ?

   (ii) Can Evan construct the foot of the altitude from A to ℓ?

10. (EGMO 2013 P5) Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q.

    Prove that $\angle ACP = \angle QCB$.

11. (USAMO 2019 P2) Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

12. (Israel 1995) Let PQ be the diameter of semicircle H. Circle O is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that $AB \perp PQ$ and is tangent to O. Prove that AC bisects $\angle PAB$.

13. (IMO 2015 P3) Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

    Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

14. (RMM 2018 P6) Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.